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SOME IDENTITIES OF ORDERED BELL NUMBERS ARISING FROM DIFFERENTIAL EQUATION

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Abstract. In this paper, we study the methods for finding the solutions of differential equations which are derived from the generating functions of ordered Bell numbers. In addition, we give some new and interesting identities of ordered Bell numbers arising from differential equations.

1. Introduction

It is well known that the ordered Bell numbers are defined by the generating function to be

$$\frac{1}{2-e^t}=\sum_{n=0}^{\infty} b_n\frac{t^n}{n!}, (\text{ see[8]}). \tag{1.1}$$

From (1.1), we note that

$$\begin{aligned}\frac{1}{2-e^t}=\frac{1}{1-(e^t-1)}&=\sum_{l=0}^{\infty}(e^t-l)^l\\&=\sum_{l=0}^{\infty}l!\sum_{n=l}^{\infty}S_2(n,l)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\big(\sum_{l=0}^{n}l!S_2(n,l)\big)\frac{t^n}{n!},\end{aligned} \tag{1.2}$$

where $S_2(n,k)$ is the Stirling number of the second kind (see [2,3,4,5,6]).

By comparing the coefficients on the both sides of (1.2), we easily get

$$b_n=\sum_{l=0}^{n}l!S_2(n,l), (\text{see [6, 8]}). \tag{1.3}$$

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For $r \in \mathbb{N}$, we define the higher-order ordered Bell numbers which are given by the generating function to be

$$\left(\frac{1}{2-e^t}\right)^r = \sum_{n=0}^{\infty} b_n^{(r)} \frac{t^n}{n!}. \tag{1.4}$$

Thus by (1.4), we get

$$\begin{aligned}\left(\frac{1}{2-e^t}\right)^r &= \underbrace{\left(\frac{1}{2-e^t}\right) \times \cdots \times \left(\frac{1}{2-e^t}\right)}_{r-\text{times}} \\ &= \Big(\sum_{l_1=0}^{\infty} b_{l_1} \frac{t^{l_1}}{l_1!}\Big) \times \cdots \times \Big(\sum_{l_r=0}^{\infty} b_{l_r} \frac{t^{l_r}}{l_r!}\Big) \\ &= \sum_{n=0}^{\infty} \Big(\sum_{l_1+\cdots+l_r=n} \frac{b_{l_1} \cdots b_{l_r}}{l_1! \cdots l_r!} n!\Big) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \Big(\sum_{l_1+\cdots+l_r=n} \binom{n}{l_1, \cdots, l_r} b_{l_1} \cdots b_{l_r}\Big) \frac{t^n}{n!}.\end{aligned} \tag{1.5}$$

By (1.5), we get

$$b_n^{(r)} = \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1, \cdots, l_r} b_{l_1} \cdots b_{l_r}. \tag{1.6}$$

Since,

$$\begin{aligned}\left(\frac{1}{2-e^t}\right)^r &= \Big(1-(e^t-1)\Big)^{-r} \\ &= \sum_{l=0}^{\infty} \binom{-r}{l} (e^t-1)^l (-1)^l \\ &= \sum_{l=0}^{\infty} \binom{r+l-1}{l} (e^t-1)^l \\ &= \sum_{l=0}^{\infty} \binom{r+l-1}{l} l! \sum_{n=l}^{\infty} S_2(n,l) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \Big(\sum_{l=0}^{n} \binom{r+l-1}{l} l! S_2(n,l)\Big) \frac{t^n}{n!}.\end{aligned} \tag{1.7}$$

From (1.6) and (1.7), we have

$$\begin{aligned} b_n^{(r)} &= \sum_{l=0}^{n} \binom{r+l-1}{l} l! S_2(n,l) \\ &= \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1,\cdots,l_r} b_{l_1}\cdots b_{l_r}. \end{aligned} \tag{1.8}$$

Recently, several authors have studied linear or non-linear differential equations which are derived from the generating functions of special polynomials and they give some new and interesting identities for the special polynomials arising from linear or non-linear differential equations(see [1-7]). In this paper, we develop the methods for doing find the solutions of differential equations which are derived from the generating functions of ordered Bell numbers. In addition, we give some new and interesting identities of ordered Bell numbers arising from differential equations.

2. Some identities of ordered Bell numbers arising from differential equations

Let

$$F = F(t) = \frac{1}{2-e^t}. \tag{2.1}$$

Then, by (2.1), we get

$$\begin{aligned} F^{(1)} = \frac{d}{dt}F(t) &= \frac{e^t}{(2-e^t)^2} \\ &= \frac{e^t-2+2}{(2-e^t)^2} \\ &= \frac{-(2-e^t)}{(2-e^t)^2} + \frac{2}{(2-e^t)^2} \\ &= -F + 2F^2. \end{aligned} \tag{2.2}$$

From (2.2), we have

$$\begin{aligned} F^{(2)} &= \frac{d}{dt}F^{(1)} = \left(\frac{d}{dt}\right)^2 F(t) \\ &= -F^{(1)} + 2^2 F F^{(1)} \\ &= -(-F+2F^2) + 2^2 F(-F+2F^2) \\ &= (-1)^2 F + (-1)(2+2^2)F^2 + 2^3 F^3. \end{aligned} \tag{2.3}$$

Taking derivative on the both sides of (2.3), we have

$$\begin{aligned} F^{(3)} &= \frac{d}{dt}F^{(2)} = \left(\frac{d}{dt}\right)^3 F(t) \\ &= (-1)^2 F^{(1)} + (-1)(2+2^2)2FF^{(1)} + 2^3 3F^2F^{(1)} \\ &= (-1)^2(-F+2F^2) + (-1)(2+2^2)2F(-F+2F^2) \\ &+ 2^3 3F^2(-F+2F^2) \\ &= (-1)^3F + (-1)^2(2+2^2+2^3)F^2 + (-1)3\cdot 2^4F^3 + 3\cdot 2^4F^4. \end{aligned} \tag{2.4}$$

Continuing this process, we get

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t) \\ &= \sum_{k=0}^{N}(-1)^{N-k}a_k(N)F^{k+1}, \ \ (N\in\mathbb{N}), \end{aligned} \tag{2.5}$$

where

$$F^{k+1} = \underbrace{F\times F\times\cdots\times F}_{k+1-\text{times}},$$

Now, we take derivative with respect to t on the both sides of (2.5).
Then we have

$$\begin{aligned} F^{(N+1)} &= \sum_{k=0}^{N}(-1)^{N-k}a_k(N)(k+1)F^kF^{(1)} \\ &= \sum_{k=0}^{N}(-1)^{N-k}a_k(N)(k+1)F^k(-F+2F^2) \\ &= \sum_{k=0}^{N}(-1)^{N-k+1}a_k(N)(k+1)F^{k+1} + 2\sum_{k=0}^{N}(-1)^{N-k}a_k(N)(k+1)F^{k+2} \\ &= \sum_{k=0}^{N}(-1)^{N-k+1}a_k(N)(k+1)F^{k+1} + 2\sum_{k=1}^{N+1}(-1)^{N-k+1}a_{k-1}(N)kF^{k+1} \\ &= \sum_{k=1}^{N}(-1)^{N-k+1}\left\{(k+1)a_k(N)+2ka_{k-1}(N)\right\}F^{k+1} \\ &+ (-1)^{N+1}a_0(N)F + 2a_N(N)\cdot(N+1)F^{N+2}. \end{aligned} \tag{2.6}$$

By replacing N by $N+1$ in (2.5), we get

$$\begin{aligned} F^{(N+1)} &= \sum_{k=0}^{N+1}(-1)^{N-k+1}a_k(N+1)F^{k+1} \\ &= \sum_{k=1}^{N}(-1)^{N-k+1}a_k(N+1)F^{k+1} \\ &+ (-1)^{N+1}a_0(N+1)F + a_{N+1}^{(N+1)}F^{N+2}. \end{aligned} \tag{2.7}$$

Comparing the coefficients on the both sides of (2.6) and (2.7), we have

$$a_0(N+1) = a_0(N), \tag{2.8}$$

$$a_{N+1}(N+1) = 2(N+1)a_N(N) \tag{2.9}$$

and

$$a_k(N+1) = (k+1)a_k(N) + 2ka_{k-1}(N) \quad \text{where } 1 \le k \le N. \tag{2.10}$$

From (2.2), we note that

$$a_0(1) = 1 \text{ and } a_1(1) = 2. \tag{2.11}$$

From (2.8), we have

$$a_0(N+1) = a_0(N) = a_0(N-1) = \cdots = a_0(1) = 1, \tag{2.12}$$

and by (2.9), we have

$$\begin{aligned} a_{N+1}(N+1) = 2(N+1)a_N(N) &= 2(N+1)\big(2Na_{N-1}(N-1)\big) \\ &= 2(N+1)2N\cdots 2\cdot 2a_1(1) \\ &= 2^{N+1}(N+1)!. \end{aligned} \tag{2.13}$$

For $k=1$ in (2.10), we have

$$\begin{aligned} a_1(N+1) &= 2a_1(N) + 2a_0(N) \\ &= 2\big(2a_1(N-1) + 2a_0(N-1)\big) + 2a_0(N) \\ &= 2^2a_1(N-1) + 2^2a_0(N-1) + 2a_0(N) \\ &= \cdots \\ &= 2^Na_1(1) + 2^Na_0(1) + \cdots + 2^1a_0(N) \\ &= 2^{N+1} + 2^N + \cdots + 2 \\ &= \sum_{i_1=1}^{N+1} 2^{i_1}. \end{aligned} \tag{2.14}$$

For $k=2$, we have

$$\begin{aligned} a_2(N+1) &= 3a_2(N) + 2^2 a_1(N) \\ &= 3\big(3a_2(N-1) + 2^2 a_1(N-1)\big) + 2^2 a_1(N) \\ &= 3^2 a_2(N-1) + 3\cdot 2^2 a_1(N-1) + 2^2 a_1(N) \\ &= \cdots \\ &= 3^{N-1} a_2(2) + 3^{N-2} 2^2 a_1(2) + \cdots + 2^2 a_1(N) \\ &= 3^{N-1} 2^2 a_1(1) + 3^{N-2} 2^2 a_1(2) + \cdots + 2^2 a_1(N) \\ &= 2^2 \sum_{i_2=0}^{N-1} 3^{i_2} a_1(N-i_2). \end{aligned} \tag{2.15}$$

By (2.14) and (2.15), we get

$$\begin{aligned} a_2(N+1) &= 2^2 \sum_{i_2=0}^{N-1} 3^{i_2} \sum_{i_1=1}^{N-i_2} 2^{i_1} \\ &= 2\cdot 2! \sum_{i_2=0}^{N-1} \sum_{i_1=1}^{N-i_2} 3^{i_2} 2^{i_1}. \end{aligned} \tag{2.16}$$

For $k=3$ in (2.10), we have

$$\begin{aligned} a_3(N+1) &= 4a_3(N) + 2\cdot 3a_2(N) \\ &= 4\big(4a_3(N-1) + 2\cdot 3a_2(N-1)\big) + 2\cdot 3a_2(N) \\ &= 4^2 a_3(N-1) + 4\cdot 2\cdot 3a_2(N-1) + 2\cdot 3a_2(N) \\ &= \cdots \\ &= 4^{N-2} a_3(3) + 4^{N-3} 2\cdot 3a_2(3) + \cdots + 2\cdot 3a_2(N) \\ &= 4^{N-2} 2\cdot 3a_2(2) + 4^{N-3} 2\cdot 3a_2(3) + \cdots + 2\cdot 3a_2(N) \\ &= 2\cdot 3 \sum_{i_3=0}^{N-2} 4^{i_3} a_2(N-i_3) \\ &= 2\cdot 3 \sum_{i_3=0}^{N-2} 4^{i_3} 2\cdot 2! \sum_{i_2=0}^{N-i_3-2} \sum_{i_1=1}^{N-i_3-i_2-1} 3^{i_2} 2^{i_1} \\ &= 2^2 3! \sum_{i_3=0}^{N-2} \sum_{i_2=0}^{N-i_3-2} \sum_{i_1=1}^{N-i_3-i_2-1} 4^{i_3} 3^{i_2} 2^{i_1}. \end{aligned} \tag{2.17}$$

Continuing this process, for $1 \leq k \leq N$, we have

$$\begin{aligned} & a_k(N+1) \\ &= 2^{k-1}k! \sum_{i_k=0}^{N-k+1} \sum_{i_{k-1}=0}^{N-k+1-i_k} \cdots \sum_{i_1=1}^{N-k+2-i_k-\cdots-i_2} 2^{i_1} \cdots k^{i_{k-1}}(k+1)^{i_k}. \end{aligned} \tag{2.18}$$

Therefore, we obtain the following differential equations.

Theorem 2.1. *Let $N \in \mathbb{N}$. Then the following differential equations,*

$$F^{(N)} = \sum_{k=0}^{N} (-1)^{N-k} a_k(N) F^{k+1},$$

have a solution $F = F(t) = \frac{1}{2-e^t}$, where

$$\begin{aligned} & a_0(N) = 1, \;\; a_N(N) = 2^N N! \; \textit{and} \\ & a_k(N) = 2^{k-1}k! \sum_{i_k=0}^{N-k} \sum_{i_{k-1}=0}^{N-k-i_k} \cdots \sum_{i_1=1}^{N-k+1-i_k-\cdots-i_2} 2^{i_1} \cdots k^{i_{k-1}}(k+1)^{i_k}. \end{aligned}$$

By (1.1) and (2.1), we easily get

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N \left(\frac{1}{2-e^t}\right) \\ &= \sum_{n=0}^{\infty} b_n \frac{1}{n!} \left(\frac{d}{dt}\right)^N t^n \\ &= \sum_{n=0}^{\infty} b_{n+N} \frac{t^n}{n!}. \end{aligned} \tag{2.19}$$

From (1.4), (1.8) and Theorem 1.1, we have

$$\begin{aligned} F^{(N)} &= \sum_{k=0}^{N}(-1)^{N-k}a_k(N)F^{k+1} \\ &= \sum_{k=0}^{N}(-1)^{N-k}a_k(N)\Big(\frac{1}{2-e^t}\Big)^{k+1} \\ &= \sum_{k=0}^{N}(-1)^{N-k}a_k(N)\sum_{n=0}^{\infty}b_n^{(k+1)}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{N}(-1)^{N-k}a_k(N)b_n^{(k+1)}\right\}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{N}(-1)^{N-k}a_k(N)\sum_{l=0}^{n}\binom{k+l}{l}l!S_2(n,l)\right\}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{N}\sum_{l=0}^{n}(-1)^{N-k}a_k(N)\binom{k+l}{l}l!S_2(n,l)\right\}\frac{t^n}{n!}. \end{aligned} \tag{2.20}$$

By comparing the coefficients on the both sides of (2.19) and (2.20), we get

$$b_{n+N} = \sum_{k=0}^{N}\sum_{l=0}^{n}(-1)^{N-k}a_k(N)\binom{k+l}{l}l!S_2(n,l). \tag{2.21}$$

Now, we consider the inversion formula of Theorem 2.1. From (2.2), we have

$$2F^2 = F + F^{(1)}. \tag{2.22}$$

Take the derivative on the both sides of (2.22), we get

$$2\cdot 2FF^{(1)} = F^{(1)} + F^{(2)}, \tag{2.23}$$

and, from (2.22), we have

$$2\cdot 2F(-F+2F^2) = F^{(1)} + F^{(2)}. \tag{2.24}$$

Thus, by (2.24), we get

$$\begin{aligned} 2^2 2!F^3 &= 2\cdot 2F^2 + F^{(1)} + F^{(2)} \\ &= 2(F+F^{(1)}) + F^{(1)} + F^{(2)} \\ &= 2F + 3F^{(1)} + F^{(2)}. \end{aligned} \tag{2.25}$$

Taking derivative on the both sides of (2.25), we have

$$2^2 2!\cdot 3F^2F^{(1)} = 2F^{(1)} + 3F^{(2)} + F^{(3)}, \tag{2.26}$$

and, by (2.22), we get

$$2^2 2! \cdot 3F^2(-F+2F^2) = 2F^{(1)} + 3F^{(2)} + F^{(3)}. \tag{2.27}$$

By (2.25) and (2.27) , we get

$$\begin{aligned} 2^3 3! F^4 &= 3 \cdot 2^2 \cdot 2! F^3 + 2F^{(1)} + 3F^{(2)} + F^{(3)} \\ &= 3(2F + 3F^{(1)} + F^{(2)}) + 2F^{(1)} + 3F^{(2)} + F^{(3)} \\ &= 6F + 11F^{(1)} + 6F^{(2)} + F^{(3)}. \end{aligned} \tag{2.28}$$

Continuing this process, we have

$$2^N N! F^{N+1} = \sum_{k=0}^{N} a_k(N) F^{(k)}. \tag{2.29}$$

Take the derivative on the both sides of (2.29) with respect to t.
Then we have

$$2^N N!(N+1) F^N F^{(1)} = \sum_{k=0}^{N} a_k(N) F^{(k+1)}. \tag{2.30}$$

From (2.22) and (2.30), we note that

$$2^N N!(N+1) F^N(-F+2F^2) = \sum_{k=0}^{N} a_k(N) F^{(k+1)}. \tag{2.31}$$

By using (2.29) and (2.31), we see that

$$\begin{aligned} 2^{N+1}(N+1)! F^{N+2} &= \sum_{k=0}^{N} a_k(N) F^{(k+1)} + (N+1) 2^N N! F^{N+1} \\ &= \sum_{k=0}^{N} a_k(N) F^{(k+1)} + (N+1) \sum_{k=0}^{N} a_k(N) F^{(k)} \\ &= \sum_{k=1}^{N+1} a_{k-1}(N) F^{(k)} + (N+1) \sum_{k=0}^{N} a_k(N) F^{(k)} \\ &= \sum_{k=1}^{N} a_{k-1}(N) F^{(k)} + (N+1) \sum_{k=1}^{N} a_k(N) F^{(k)} \\ &+ a_N(N) F^{(N+1)} + (N+1) a_0(N) F \\ &= \sum_{k=1}^{N} \big(a_{k-1}(N) + (N+1) a_k(N)\big) F^{(k)} \\ &+ a_N(N) F^{(N+1)} + (N+1) a_0(N) F. \end{aligned} \tag{2.32}$$

By replacing N by N+1 in (2.29), we get

$$2^{N+1}(N+1)!F^{N+2} = \sum_{k=0}^{N+1} a_k(N+1)F^{(k)}. \tag{2.33}$$

Comparing the coefficients on the both sides of (2.32) and (2.33), we have

$$a_0(N+1) = (N+1)a_0(N), \quad a_{N+1}(N+1) = a_N(N), \tag{2.34}$$

and

$$a_k(N+1) = (N+1)a_k(N) + a_{k-1}(N), \quad (1 \le k \le N). \tag{2.35}$$

By (2.22) and (2.29), we get

$$2^1 1!F^2 = F + F^{(1)} = a_0(1)F + a_1(1)F^{(1)}. \tag{2.36}$$

From (2.36), we have

$$a_0(1) = 1 \ \ and \ \ a_1(1) = 1. \tag{2.37}$$

Thus, by (2.34) and (2.37), we get

$$\begin{aligned} a_0(N+1) &= (N+1)a_0(N) = (N+1)Na_0(N-1) = \cdots \\ &= (N+1)N\cdots 2a_0(1) = (N+1)N\cdots 2\cdot 1 = (N+1)!, \end{aligned} \tag{2.38}$$

and

$$a_{N+1}(N+1) = a_N(N) = a_{N-1}(N-1) = \cdots = a_1(1) = 1. \tag{2.39}$$

For $k = 1$ in (2.35), we have

$$\begin{aligned} a_1(N+1) &= (N+1)a_1(N) + a_0(N) \\ &= (N+1)\big(Na_1(N-1) + a_0(N-1)\big) + a_0(N) \\ &= (N+1)Na_1(N-1) + (N+1)a_0(N-1) + a_0(N) \\ &= \cdots \\ &= (N+1)_N a_1(1) + (N+1)_{N-1}a_0(1) + \cdots + a_0(N) \\ &= (N+1)_N a_0(0) + (N+1)_{N-1}a_0(1) + \cdots + a_0(N) \\ &= \sum_{i_1=0}^{N} (N+1)_{N-i_1} a_0(i_1) \\ &= \sum_{i_1=0}^{N} (N+1)_{N-i_1} i_1!. \end{aligned} \tag{2.40}$$

For $k = 2$, we have

$$\begin{aligned}
a_2(N+1) &= (N+1)a_2(N) + a_1(N) \\
&= (N+1)\big(Na_2(N-1) + a_1(N-1)\big) + a_1(N) \\
&= (N+1)Na_2(N-1) + (N+1)a_1(N-1) + a_1(N) \\
&= \cdots \\
&= (N+1)_{N-1}a_2(2) + (N+1)_{N-2}a_1(2) + \cdots + a_1(N) \\
&= (N+1)_{N-1}a_1(1) + (N+1)_{N-2}a_1(2) + \cdots + a_1(N) \\
&= \sum_{i_2=0}^{N-1}(N+1)_{N-i_2-1}a_1(i_2+1) \\
&= \sum_{i_2=0}^{N-1}(N+1)_{N-i_2-1}\sum_{i_1=0}^{i_2}(i_2+1)_{i_2-i_1}i_1! \\
&= \sum_{i_2=0}^{N-1}\sum_{i_1=0}^{i_2}(N+1)_{N-i_2-1}(i_2+1)_{i_2-i_1}i_1!.
\end{aligned} \tag{2.41}$$

For $k = 3$ in (2.35), we have

$$\begin{aligned}
a_3(N+1) &= (N+1)a_3(N) + a_2(N) \\
&= (N+1)\big(Na_3(N-1) + a_2(N-1)\big) + a_2(N) \\
&= (N+1)Na_3(N-1) + (N+1)a_2(N-1) + a_2(N) \\
&= \cdots \\
&= (N+1)_{N-2}a_3(3) + (N+1)_{N-3}a_2(3) + \cdots + a_2(N) \\
&= (N+1)_{N-2}a_2(2) + (N+1)_{N-3}a_2(3) + \cdots + a_2(N) \\
&= \sum_{i_3=0}^{N-2}(N+1)_{N-i_3-2}a_2(i_3+2) \\
&= \sum_{i_3=0}^{N-2}(N+1)_{N-i_3-2}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{i_2}(i_3+2)_{i_3-i_2}(i_2+1)_{i_2-i_1}i_1! \\
&= \sum_{i_3=0}^{N-2}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{i_2}(N+1)_{N-i_3-2}(i_3+2)_{i_3-i_2}(i_2+1)_{i_2-i_1}i_1!.
\end{aligned} \tag{2.42}$$

Continuing this process, for $1 \leq k \leq N$, we have,

$$\begin{aligned} a_k(N+1) = & \sum_{i_k=0}^{N-k+1} \sum_{i_{k-1}=0}^{i_k} \cdots \sum_{i_1=0}^{i_2} (N+1)_{N-i_k-k+1}(i_k+k-1)_{i_k-i_{k-1}} \\ & \times \cdots \times (i_2+1)_{i_2-i_1} i_1! \\ = & \sum_{i_k=0}^{N-k+1} \sum_{i_{k-1}=0}^{i_k} \cdots \sum_{i_1=0}^{i_2} \frac{(N+1)!}{\prod_{j=1}^{k}(i_j+j)}. \end{aligned} \tag{2.43}$$

Therefore, we obtain the following theorem.

Theorem 2.2. (**Fundamental identity**) *For $N \in \mathbb{N}$, we have*

$$2^N N! F^{N+1} = \sum_{k=0}^{N} a_k(N) F^{(k)},$$

where $a_0(N) = N!$, $a_N(N) = 1$, *and*

$$a_k(N) = \sum_{i_k=0}^{N-k} \sum_{i_{k-1}=0}^{i_k} \cdots \sum_{i_1=0}^{i_2} \frac{N!}{\prod_{j=1}^{k}(i_j+j)}.$$

From Theorem 2.2, we note that

$$\begin{aligned} 2^N N! F^{N+1} &= 2^N N! \Big(\frac{1}{2-e^t}\Big)^{N+1} \\ &= 2^N N! \sum_{n=0}^{\infty} b_n^{(N+1)} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} 2^N N! b_n^{(N+1)} \frac{t^n}{n!}, \end{aligned} \tag{2.44}$$

and

$$\begin{aligned} 2^N N! F^{N+1} &= \sum_{k=0}^{N} a_k(N) F^{(k)} \\ &= \sum_{k=0}^{N} a_k(N) \Big(\frac{d}{dt}\Big)^k \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{N} a_k(N) b_{n+k} \Big) \frac{t^n}{n!}. \end{aligned} \tag{2.45}$$

By (1.3), (2.44) and (2.45), we easily get

$$\begin{aligned}2^N N! b_n^{(N+1)} &= \sum_{k=0}^{N} a_k(N) b_{n+k} \\ &= \sum_{k=0}^{N} a_k(N) \sum_{l=0}^{n+k} l! S_2(n+k,l) \\ &= \sum_{k=0}^{N} \sum_{l=0}^{n+k} a_k(N) l! S_2(n+k,l).\end{aligned} \tag{2.46}$$

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